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Numerical Solutions of Exterior Problems with the Reduced Wave Equation*

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A new technique for numerically solving the reduced wave equation on exterior domains is presented. The method is basically a relaxation scheme. It is general enough to handle both inhomogeneous and nonlinear indices of refraction. Although the convergence is slow, the technique is tested on two classical problems: the scattering of a plane wave off a metal cylinder and off a metal sphere. The results are in good qualitative agreement with previously calculated values. In particular, the numerical solutions exhibit the correct diffractive effects at moderate frequencies.

1. THE METHOD

We are concerned here with a numerical method for determining $U(\mathbf{x})$, the solution to

$$\Delta U + \omega^2 n(\mathbf{x}) U = 0, \tag{1}$$

in an exterior region, i.e., a region containing the point at ∞. The index of refraction, $n(\mathbf{x})$, is some smooth function equal to 1 at ∞ and is defined in N-dimensional Euclidean space or some subdomain. The parameter $\omega^2 = (ka)^2$, where k is the wavenumber of the incident wave and a is a characteristic dimension of the scatterer. Since we are primarily concerned with the effect of the scatterer on an incident plane wave, we decompose the "total field" $U(\mathbf{x})$ as follows:

$$U = e^{i\omega x} + u(\mathbf{x}). \tag{2}$$

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That is, the total field is the sum of an incident plane wave and a scatterd field $u(\mathbf{x})$ that satisfies

$$\Delta u + \omega^2 n(\mathbf{x}) u + \omega^2 [n(\mathbf{x}) - 1] \, e^{i\omega x} = 0. \tag{3}$$

The scattered function $u(\mathbf{x})$ satisfies the Sommerfeld radiation condition at ∞,

$$\frac{\partial}{\partial r} u - i\omega u + \frac{(N-1)}{2r} u \sim 0 \qquad \text{as} \quad r \to \infty, \tag{4}$$

or equivalently,

$$u \sim \sigma(\theta) \frac{e^{i\omega r}}{r^{(N-1)/2}} \left(1 + O\left(\frac{1}{r}\right)\right) \qquad \text{as} \quad r \to \infty, \tag{4'}$$

where θ represents the angular variables. Here the term $(1 + O(1/r))$ is a power series in $1/r$.

It is important to note that the corresponding time-dependent incident wave must have the time dependence $\exp(-i\omega t)$ for (4) and (4′) to be valid (i.e., the wave must be outgoing). This implies that the incident plane wave is coming from $-\infty$ in the x_1 direction and is moving "from left to right."

We next set

$$u = v e^{i\omega r} \tag{5}$$

and find from (3) that v must satisfy

$$\Delta v + 2i\omega(\nabla v \cdot \nabla r) + i\omega(\Delta r) v + \omega^2(n(x) - 1) v = f, \tag{6}$$

where

$$f = \omega^2(1 - n(x)) \, e^{i\omega x - i\omega r}. \tag{7}$$

Combining (4′) and (5), we deduce that v satisfies

$$\frac{\partial}{\partial r} v + \frac{(N-1)}{2r} v \sim 0 \qquad \text{as} \quad r \to \infty, \tag{8}$$

or equivalently,

$$v \sim \frac{\sigma(\theta)}{r^{(N-1)/2}} \left(1 + O\left(\frac{1}{r}\right)\right) \qquad \text{as} \quad r \to \infty. \tag{8'}$$

We rewrite (6) and (8′) in operator notation as

$$Lv = f, \qquad \frac{\partial}{\partial r} (v r^{(N-1)/2}) = O(r^{-2}) \qquad \text{as} \quad r \to \infty. \tag{9}$$

The method for solving this equation numerically is to find a hyperbolic time-dependent equation with the property that as the time t tends to infinity the solution tends to the solution of (9). How to find this equation is discussed in [2] for general L. Here we show directly that the equation

$$(M(p))_t = L(p) - f \tag{10}$$

has the desired property if we require

$$M = \frac{\partial}{\partial r} + \frac{N-1}{r} \tag{11}$$

and

$$\frac{\partial}{\partial r}(pr^{(N-1)/2}) = O(r^{-2}) \qquad \text{as} \quad r \to \infty.$$

We next set $vr^{(N-1)/2} = w$ and note that (9) becomes

$$0 = L(wr^{-(N-1)/2}) - f, \qquad \frac{\partial w}{\partial r} = O(r^{-2}) \qquad \text{as} \quad r \to \infty. \tag{12}$$

The appropriate time-dependent problem for $\tilde{W}$ is

$$2\tilde{W}_{rt} = r^{(N-1)/2}[L(\tilde{W}r^{-(N-1)/2}) - f] \tag{13}$$

with

$$\tilde{W}_r \sim O(r^{-2}) \qquad \text{as} \quad r \to \infty. \tag{14}$$

We now consider (13) and (14) for the two practical situations $N = 2$ and $N = 3$.

1.1. *Case* I: $N = 2$

$$2\tilde{W}_{rt} = \tilde{W}_{rr} + 2i\omega\tilde{W}_r + \frac{1}{r^2}\tilde{W}_{\theta\theta} + \left[\omega^2(n-1) + \frac{1}{4r^2}\right]\tilde{W} - r^{1/2}f, \tag{15}$$

$$\frac{\partial}{\partial r}\tilde{W} = O\left(\frac{1}{r^2}\right) \qquad \text{as} \quad r \to \infty. \tag{16}$$

Finally, we set $\tilde{W} = e^{i\omega t}W$ and obtain

$$2W_{rt} = W_{rr} + \frac{1}{r^2}W_{\theta\theta} + \left[\omega^2(n-1) + \frac{1}{4r^2}\right]W - r^{1/2}fe^{-i\omega t}, \tag{17}$$

$$W_r \sim 0 \qquad \text{as} \quad r \to \infty, \tag{18}$$

$$U = e^{i\omega x} + W(r, \theta, t)\, e^{i\omega(t+r)}/r^{1/2}. \tag{19}$$

These are the equations to be solved numerically.[1]

We check the decay property here. The system for $\tilde{W}_t = W'$ is

$$\begin{aligned} &2W'_{rt} = W'_{rr} + 2i\omega W'_r + \frac{1}{r^2}W'_{\theta\theta} + \left(\omega^2(n-1) + \frac{1}{4r^2}\right)W', \\ &W'_r = O(1/r^2) \qquad \text{as} \quad r \to \infty. \end{aligned} \tag{20}$$

[1] Equation (17) is the wave equation with a transformation of variables.

We multiply this equation by rW'_r, integrate the real part over the spatial domain $\mathscr{D}$ of the solution, use $W'_r = O(1/r^2)$, W' bounded, and obtain

$$\frac{\partial}{\partial t}\iint |rW'_r|^2\,dr\,d\theta = \frac{1}{2}\int_{\partial\mathscr{D}} \{r\,|W'_r|^2 - r^{-1}|W'_\theta|^2 + \omega^2(n-1)\,r\,|W'|^2\}\,d\theta$$
$$- \int_{\partial\mathscr{D}} \operatorname{Re}\overline{W'_\theta}W'_r\,ds - \iint_{\mathscr{D}} (\omega^2(n-1)\,r)_r\,|W'|^2\,dr\,d\theta$$
$$- \frac{1}{2}\iint_{\mathscr{D}} \left(|W'_r|^2 - \frac{1}{2r}\operatorname{Re}\overline{W'_r}W'\right)dr\,d\theta.$$

The last integral is rewritten as

$$\iint_{\mathscr{D}} \left\{\left|W'_r - \frac{1}{2r}W'\right|^2 + \frac{1}{2r}\operatorname{Re}W'_rW' - \frac{1}{4r^2}|W'|^2\right\}dr\,d\theta$$
$$= \iint_{\mathscr{D}} \left|W'_r - \frac{1}{2r}W'\right|^2 dr\,d\theta + \int_{\partial\mathscr{D}} \frac{1}{4r}|W'|^2\,d\theta,$$

so that

$$\frac{\partial}{\partial t}\iint |rW'_r|^2\,dr\,d\theta$$
$$= \frac{1}{2}\int_{\partial\mathscr{D}} \left\{r\,|W'_r|^2 - r^{-1}|W'_\theta|^2 + \left(\frac{1}{4r} + \omega^2(n-1)\,r\right)|W'|^2\right\}d\theta$$
$$- \int_{\partial\mathscr{D}} \operatorname{Re}\overline{W'_\theta}W'_r\,dr - \iint_{\mathscr{D}} \Big\{(\omega^2(n-1)\,r)_r\,|W'|^2$$
$$+ \frac{1}{2}\left|W'_r - \frac{1}{2r}W'\right|^2\Big\}\,dr\,d\theta. \tag{21}$$

For the moment we assume conditions so that the integrals are negative.

If we had had $2r^{-1}W'_r$ on the left-hand side of (20), we would now have

$$\frac{dI}{dt} \leqslant -\frac{1}{2}I, \qquad \text{where} \quad I = \iint |W'_r|^2\,dr\,d\theta,$$

and hence would obtain exponential decay. This, however, would raise numerical difficulties. The leading operator would be

$$2\frac{\partial^2}{\partial r\,\partial t} - r\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial^2}{\partial\theta^2}$$

instead of

$$2\frac{\partial^2}{\partial r\,\partial t} - \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Dropping the θ-differentiation for convenience, one sees that in the first case the characteristic speeds are the roots γ/β of $2\beta\gamma - r\beta^2 = 0$ and $2\beta\gamma - \beta^2 = 0$, respectively, i.e., ∞ and $r/2$ or ∞ and $\frac{1}{2}$. In any difference scheme we may introduce later we must for stability at least pick up data at a previous time over a sufficiently large interval. (However, in this case we cannot, as in [1], add a term W_t to obtain stability.) These data are all on a characteristic surface (t = constant) and hence the rate $\Delta r/\Delta t$ is at least determined by the *second* characteristic speed. Thus the marginal rates are $\Delta r/\Delta t > r/2$ and $\Delta r/\Delta t > 1/2$, respectively. The first is bad for large distances. The second is uniform and was the one used. The best scheme would probably involve $g(r)\, W_{rt}$ as the left-hand operator, with g adjusted to the particular problem.

But returning to the question of decay we have $\iint r \mid W_r' \mid^2 dr\, d\theta$ decreases monotonely, and since it is positive its time derivative goes to zero; hence the integrals on the right go to zero, and thus by their fixed sign W', W_r', W_θ' all go to zero in L^2. We may assume smoothness, and thus we have W_t and its derivatives go to zero. From (15) we see that $r^{1/2}\tilde{W}$ tends to the steady solution of (9).

It remains only to determine what makes the integrals of (21) negative definite.

(A) We assume that the origin lies inside $\partial\mathscr{D}$ and that W' satisfies the Dirichlet condition, that is, W is prescribed on $\partial\mathscr{D}$. Then using $W_\theta'\, d\theta + W_r'\, dr = 0$ on $\partial\mathscr{D}$ as well as $W' = 0$, we have in (21) the boundary integral $\frac{1}{2}\int_{\partial\mathscr{D}} (r \mid W_r' \mid^2 + \mid W_\theta' \mid^2)\, d\theta$. Thus if $d\theta < 0$ on $\partial\mathscr{D}$, i.e., if $\partial\mathscr{D}$ is star-shaped, the boundary integral is negative as required.

(B) If the boundary reduces to $r = 0$ (no object) since $W' \sim r^{1/2}$, $W_\theta' \sim r^{3/2}$, $d\theta < 0$, the boundary integral is again negative as required.

(C) The condition on the index of refraction to make the volume integral have the right sign is

$$((n-1)r)_r > 0.$$

It should be remarked that many numerical examples work even if this condition is violated. It is not, however, a sharp condition. There will be decay (see [2]) if there are no trapped rays. If $n = n(r)$ the trapped rays are circles and cannot occur if $(nr^2)_r > 0$ or $((n-1)r)_r > (-1-n)$. If this condition is violated the solution to the original problem may be close to an outgoing solution corresponding to a complex eigenvalue of small imaginary part.

1.2. *Case* II: $N = 3$

Introducing polar coordinates, we write (13), (14) as

$$2\tilde{W}_{rt} = \tilde{W}_{rr} + 2i\omega\tilde{W}_r + (1/r^2)\,\mathscr{B}(\tilde{W}) + \omega^2(n-1)\tilde{W} - rf, \tag{22}$$

$$\tilde{W}_r = O(1/r^2) \qquad \text{as} \quad r \to \infty. \tag{23}$$

The reader can check the decay to a steady state in a somewhat simpler way and with the same conditions on $\partial\mathscr{D}$ and n. Here $\mathscr{B}$ is the Laplace–Beltrami operator.

Finally, we set $\tilde{W} = \exp(i\omega t)W$ and obtain

$$2W_{rt} = W_{rr} + (1/r^2)\mathscr{B}(W) + \omega^2(n-1)W - rfe^{-i\omega t}, \tag{24}$$

$$W_r \to 0 \quad \text{as} \quad r \to \infty, \tag{25}$$

$$U = e^{i\omega x} + W(r, \theta, t)\, e^{i\omega(t+r)}/r, \tag{26}$$

$$\mathscr{B}(W) = \frac{1}{\sin\theta}\frac{1}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 W}{\partial\phi^2}, \tag{27}$$

and f is given by (7).

2. The Two-Dimensional Problem

To complete the formulation of the time-dependent problem before introducing the difference equation for (17) we must prescribe more data. First, since (17) has t = constant as a characteristic surface, we must give $W(r, \theta, t)$ exactly one initial datum, i.e., its value

$$W(r, \theta, 0) = Q(r, \theta), \tag{28}$$

where Q is arbitrary. Our experimental observation is that the initial datum is "swept away" after a characteristic time. Second, we must give $W(r, \theta, t)$ a compatible value on the boundary of the scatterer or a regularity condition at $r = 0$. We distinguish two cases that were computed:

Reflecting cylinder. The scatterer is a conducting cylinder $r = 1$; i.e., the total field $U = 0$ vanishes on the boundary. We take the circular cylinder with the origin at its center for simplicity in the numerical scheme. From (19) we find that

$$W(1, \theta, t) = -\exp[i\omega(\cos\theta - t - 1)]. \tag{29}$$

Inhomogeneous cylinder. In this case the scatterer is an object occupying the region $r \leqslant 1$. This object is characterized by its nonconstant index of refraction $n(r)$. Since the total field $U(x)$ is well behaved at $r = 0$, it follows from (19) that

$$W(0, \theta, t) = 0. \tag{30}$$

The difference scheme. The grid pattern which we use is a rectangular grid in r, θ space. We obtain the difference equation for (17), with W_{jm}^n the value of the solution at $r = r_j = j\,\Delta r$ + const, $\theta = m\,\Delta\theta$ at time $t = n\,\Delta t$. We replace the Laplacian by the obvious differences,

$$W_{rr} + \frac{1}{r^2}W_{\theta\theta} \to \frac{1}{h^2}\{W_{j+1,m}^n - 2W_{j,m}^n + W_{j-1,m}^n\} + \frac{\mu^2}{h^2 r_j^2}\{W_{j,m+1}^n - 2W_{j,m}^n + W_{j,m-1}^n\}. \tag{31}$$

For the mixed derivative term we take

$$W_{rt} \to (1/\lambda h^2)[(W^{n+1}_{j+1,m} - W^{n+1}_{j-1,m}) - (W^{n}_{j+1,m} - W^{n}_{j-1,m})]. \tag{32}$$

Finally we have that

$$[\omega^2(n - 1) + 1/4r^2]\, W - r^{1/2} f e^{-i\omega t}$$
$$\to \{\omega^2(n_{j,m} - 1) + 1/4r_j^2\}\, W^n_{j,m} - r_j^{1/2} f_{j,m} e^{-i\omega t_n}. \tag{33}$$

In (31) and (32),

$$h = \Delta r, \qquad \mu = \Delta r/\Delta\theta, \qquad \lambda = \Delta t/\Delta r. \tag{34}$$

This difference scheme conserves some of the quantities that go into (17). It was too awkward to introduce a nine-point scheme that would have conserved exactly.

Combining (31)–(33) we arrive at

$$\begin{aligned} W^{n+1}_{j+1,m} &= W^{n+1}_{j-1,m} + aW^{n}_{j+1,m} - b_j W^{n}_{j,m} \\ &= cW^{n}_{j-1,m} + d_j W^{n}_{j,m+1} + e_j W^{n}_{j,m-1} - \lambda\omega^2 h^2 F^{n}_{j,m}, \end{aligned} \tag{35}$$

where

$$F^{n}_{j,m} = r_j^{1/2}(1 - n_{j,m})\, e^{i\omega(r_j \cos\theta_m - r_j - \lambda h n)}, \tag{36}$$

$$a = 1 + \lambda; \qquad c = \lambda - 1, \tag{37}$$

$$b_j = 2\lambda[1 + \mu^2/r_j^2] - \lambda h^2/4r_j^2 - \lambda\omega^2 h^2(n_{j,m} - 1), \tag{38}$$

$$d_j = \lambda\mu^2/r_j^2 = e_j\,. \tag{39}$$

The domain is taken to be $\theta \in [0, 2\pi]$, $r \in [0, R]$ or $r \in [1, R]$ depending on which case we are dealing with. Note that the index of refraction $n_{j,m}$ apperas only in the old time step. Hence, although we may be violating the decay condition C, we can always take the index to be any nonlinear function of the solution or even its spatial derivatives and update it. Our experiments were confined to taking $n - 1$ to be quadratic in the absolute value of the total field. This is the typical nonlinearity proposed for laser beams.

2.1. *The Reflecting Cylinder*

In this section we solve Eq. (35) for the case of a reflecting cylinder. The step size h is $(R - 1)/N$, where N is an even integer. Since $r \in [1, R]$, we set

$$r_1 = 1, \qquad r_j = 1 + (j - 1)h \qquad \text{for} \quad 1 \leqslant j \leqslant N + 1 \tag{40}$$

and

$$r_{N+1} = R. \tag{41}$$

With $\theta \in [0, 2\pi]$, the step size $\Delta\theta$ is $2\pi/M$, where M is any integer. We set

$$\theta_1 = 0, \qquad \theta_m = (m-1)\,\Delta\theta \qquad \text{for} \quad 1 \leqslant m \leqslant (M+1) \tag{42}$$

and

$$\theta_{M+1} = 2\pi. \tag{43}$$

Now at $r = R$ we impose the radiation condition (18) in the form

$$W^n_{N+1,m} = W^n_{N,m} \tag{44}$$

for all m and $n \geqslant 0$. The boundary condition (29) becomes

$$W^n_{1,m} = -\exp[i\omega(\cos\theta_m - n\lambda h - 1)]. \tag{45}$$

Since W is periodic in θ, we have

$$W^n_{j,M+m} = W^n_{j,m} \tag{46}$$

for $n \geqslant 0$, $1 \leqslant j \leqslant N+1$, $1 \leqslant m \leqslant M$. The initial conditions become

$$W^0_{j,m} = Q_{j,m}, \qquad 1 \leqslant j \leqslant N+1, \quad 1 \leqslant m \leqslant M+1, \tag{47}$$

and must satisfy (45) and (46). We now explain how (35) is solved *explicitly*.

Consider the case when $j = 2$, $n = 0$, and $m = 2$. From (35) we see that, except for the first term, the right-hand side is evaluated at $t = 0$ and hence is known. The first term, $W^1_{1,2}$, is given by (45), since it is the value of W on the boundary. Thus $W^1_{3,2}$ can be determined. By increasing m with $j = 2$ fixed, we obtain

$$W^1_{3,m} \qquad \text{for} \quad m = 2, 3,\ldots, M+1.$$

We next use (46) and obtain $W^1_{3,1}$. Proceeding outward in j we compute

$$W^1_{j,m} \qquad \text{for} \quad j = 1, 3, 5, 7,\ldots, (N+1); \quad 1 \leqslant m \leqslant M+1.$$

This is where the evenness of N is used. We now know W at the odd-numbered "r mesh points." To fill in the others we use (44) to find

$$W^1_{N,m} = W^1_{N+1,m}.$$

Next we solve (35) for $W^1_{j-1,m}$ obtaining $W^1_{j-1,m} = W^1_{j+1,m} + F(W^0_{i,k})$ and set $j = N-1$ to obtain

$$W^1_{N-2,m} = W^1_{N,m} + F(*) \qquad \text{for} \quad 1 \leqslant m \leqslant M+1.$$

Continuing this process we "march back" toward the boundary $r = 1$ and pick up the even-numbered mesh points, i.e.,

$$W_{j,m}^{1} \quad \text{for} \quad j = N-2, N-4, \ldots, 2; \quad 1 \leqslant m \leqslant M+1.$$

We now know W at the first time step.

The entire sweeping process is repeated until convergence has been obtained. To determine the convergence numerically, recall that for large t the scattered field u is given approximately by

$$u(r_j, \theta_j) \simeq [W_{i,j}^{n} e^{i\omega t_n}](e^{i\omega r_j}/r_j^{1/2}).$$

The bracketed term approximates the solution W of Eq. (15). Since W approaches a "steady state" for large time, the term $W_{i,j}^{n} e^{i\omega t_n}$ must become independent of n for large n. Thus the magnitude $| W_{i,j}^{n} |$ becomes independent of n. We terminate our computation when

$$\sum_{i=1}^{N+1} \sum_{j=1}^{M+1} \{| W_{i,j}^{n+1} | - | W_{i,j}^{n} |\}^2 < \epsilon$$

for some prescribed $\epsilon > 0$.

2.2. *The Inhomogeneous Medium*

The scheme here is the same as that in the previous case except for two points:

(1) $r \in [0, R]$, $h = R/N$, and $r_1 = 0$,

and from (30),

(2) $W_{1,m}^{n} = 0$ for $1 \leqslant m \leqslant M + 1$; $n \geqslant 0$.

It should be noted here that both the differential and difference equations are singular at $r = r_1 = 0$. To avoid evaluating Eq. (35) at $r_1 = 0$ we "introduced a metal cylinder of radius $2h$" about the origin by setting

$$W_{1,m}^{n} = W_{2,m}^{n} = W_{3,m}^{n} = 0; \qquad 1 \leqslant m \leqslant M+1; \quad n \geqslant 0, \tag{48}$$

and introduced an error of order $(4h)^{1/2}$.

2.3. *Numerical Experiments*

(A) The scattering of plane waves off a reflecting cylinder is a well-studied physical problem. The classical attack is to separate variables and sum the resulting Fourier series. The coefficients in this series involve both Bessel and Hankel functions —the sum is not known analytically. Various asymptotic methods are available for the cases where $ka \ll 1$ and $ka \gg 1$. These are the quasi-static method and geometrical optics method, respectively, [4]. However, when $ka = O(1)$ one must sum numerically a sufficiently large number of terms to obtain a reasonable answer. When ka starts to become large this number increases dramatically. (It was a problem similar to this which led Watson to invent his now famous transform.) Moreover, for each value of θ the sum must be recomputed. This is where computation time is consumed.

We have applied our method to this problem for two cases; $\omega = 1$ and $\omega = 5$. Our results are shown graphically in Figs. 1 and 2. In both cases we have achieved good qualitative agreement with the graphical results presented in [4]. To make the comparison more quantitative we have converted the graphical information given in [4] into tabular form. We have shown these values in Table I for the case $\omega = 5$. Our results are shown there also; the agreement is very good. The discrepancies are caused by two effects: the extrapolation of graphical data and the finite size of our numerical grid.

For both the cases of $\omega = 1$ and $\omega = 5$ we have $1 \leqslant r \leqslant 9$ with $h = 0.1$ and $N = 80$. In order to conserve computer space we made the following change in our method. Making use of the symmetry of the solution about the x axis (i.e., $W(r, \theta) =$

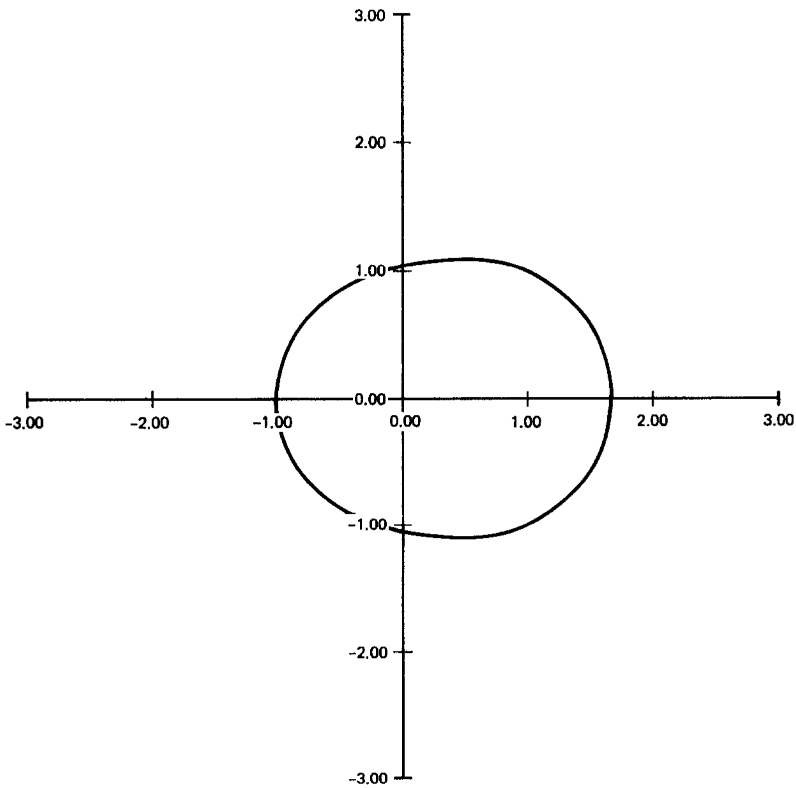

FIG. 1. The polar graph of the cross section, $(\pi\omega/2)^{1/2} S(\theta)$, for a metal cylinder with $u \sim S(\theta)$ $(e^{i\omega r}/r^{1/2})$, $\omega = ka = 1$, $\Delta r = 0.1$, $\Delta\theta = \pi/40$, and $\Delta t = 1/250$.

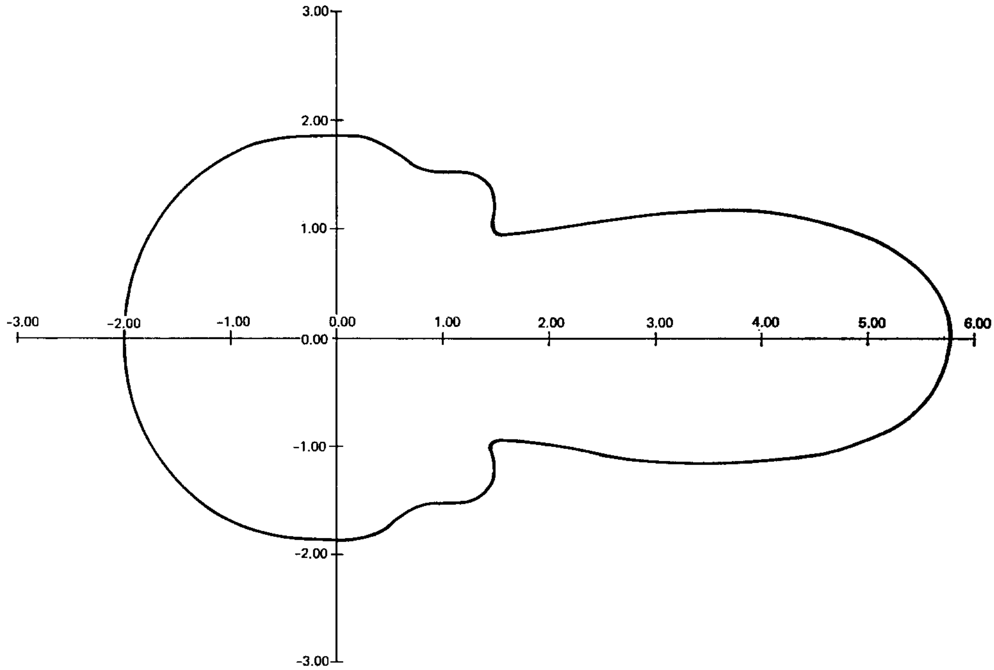

FIG. 2. Same as Fig. 1 with $\omega = 5$.

TABLE I

The cross section $(\pi\omega/2)^{1/2}S(\theta)$, for a metal cylinder with $u \sim S(\theta)(e^{i\omega r}/r^{1/2})$, $\omega = ka = 5$, $\Delta r = 0.1$, $\Delta\theta = \pi/40$, and $\Delta t = 1/250$

θ (°)	Kriegsmann and Morawetz	Bowman *et al.*
0	5.67	6.0
9	5.30	5.25
18	3.78	3.68
27	2.10	1.98
36	1.81	1.83
45	2.0	1.95
54	1.89	1.83
63	1.74	1.66
72	1.79	1.75
81	1.86	1.83
90	1.86	1.83
108	1.93	1.88
126	1.97	1.92
144	1.99	1.99
162	1.99	1.99
180	2.00	2.00

$W(r, -\theta)$, it was necessary to solve for W in the contracted region, $\theta \in [0, \pi]$; $r \in [1, 9]$. The new boundary conditions are then

$$\frac{\partial}{\partial \theta} W(r, \theta)\Big|_{\theta=0,\pi} = 0. \tag{49}$$

In terms of the difference approximation this gives

$$W^n_{j,m+1} = W^n_{j,m} \qquad \text{and} \qquad W^n_{j,1} = W^n_{j,2} \tag{49'}$$

for $n \geqslant 0$, $1 \leqslant j \leqslant (N + 1)$.

In both these cases the time was allowed to become large enough to ensure that any initial data or noise was "swept" away (see Appendix). Although our initial guess was $W = 0$ for both cases, the numerical solutions had essentially converged when n became larger than $n_{\max}$, where

$$n_{\max} = (2 \cdot 8)/\Delta t.$$

In this formula the factor 2 arises from the slope of the characteristic line, see (22). The factor 8 is the width of the numerical grid in the r direction.

(B) The scattering of plane waves by *inhomogeneous* media is a problem which has also received considerable attention. The cylinder models in some cases a plasma target (rf-heated Tokamak involving a two-dimensional pellet). The problem has been studied analytically by geometrical optics and numerically for $n = n(r)$. The same separation method is used, but now the radial eigenfunctions must be computed. If $n = n(r, \theta)$ or is nonlinear the separation-of-variables method is useless, since all modes are coupled together.

In our computer runs we have chosen for comparison

$$\begin{aligned} n(\mathbf{x}) &= p(r); && \theta \leqslant r \leqslant \tfrac{1}{2}, \\ &= 1; && \tfrac{1}{2} \leqslant r \leqslant 2, \end{aligned}$$

and as a nonlinear example

$$n(\mathbf{x}) = 1 + (p(r) - 1)\, A \mid U \mid^2,$$

where U is total field; see (1). We have run the problem with $\theta \in [0, 2\pi]$ because in general $W(r, \theta) \neq W(r, -\theta)$ unless $n(\mathbf{x})$ has this symmetry. Thus for storage reasons we were limited to $R = 2$.

In any case, we choose both a quadratic and a linear $p(r)$:

$$\begin{aligned} p &= 4r^2 && \text{for} \quad 0 \leqslant r \leqslant \tfrac{1}{2}, \\ &= 1 && \text{for} \quad \tfrac{1}{2} \leqslant r \leqslant 2, \end{aligned}$$

and

$$\begin{aligned} p &= 2(1 - y_0)r + y_0 && \text{for} \quad 0 \leqslant r \leqslant \tfrac{1}{2}, \\ &= 1 && \text{for} \quad \tfrac{1}{2} \leqslant r \leqslant 2. \end{aligned}$$

In both cases the numerical solutions converged after the characteristic time. We even adjusted y_0 so $n(0) < 0$ and the solution still converged.

We have not compared our runs with any previously tabulated results, but the numerical solutions converged and were completely independent of the initial guess.

3. The Three-Dimensional Symmetric Problem

Before introducing the difference equation for (24) we add the initial and boundary data necessary for a well-posed problem. First, the initial values are

$$W(r, \theta, \phi, 0) = Q(r, \theta). \tag{50}$$

In order to have only two space variables we assume $(\partial/\partial\phi) W = 0$. The method, however, could be applied equally well to the full three-dimensional problem, in principle. This will hold if the incident wave is a plane wave directed along $\theta = 0$.

With this incident plane wave we find that (24) reduced to

$$2W_{rt} = W_{rr} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta} W\right) + \omega^2(n-1)\, W - rfe^{-i\omega t} \tag{51}$$

with f given by (7), the behavior at ∞ by (25), and the total field by (26).

Now since U is regular at the origin we have

$$W(0, \theta, t) = 0. \tag{52}$$

Furthermore, from (51) we note the singular term $\cot\theta W_\theta$. But U is regular at $\theta = 0$, π so that

$$\frac{\partial}{\partial\theta} W = 0 \qquad \text{for} \quad \theta = 0, \pi. \tag{53}$$

Finally, if we consider the scattering of a reflecting sphere at $r = 1$, we have $U = 0$ there or

$$W(1, \theta, t) = -\exp[i\omega(\cos\theta - t - 1)]. \tag{54}$$

We now give the difference equation for (51) with $W^n_{j,m}$ defined as in the previous cases.

For the terms $W_{rr} + (1/r^2)\, W_{\theta\theta}$ we use the difference expression given in (31). For the term $2W_{rt}$ we use (32). Note there is no term $1/4r^2$, but instead the term $\cot\theta W_\theta$ for which we set

$$W_\theta \cot\theta \to \frac{\cot\theta_m}{2r_j^2(\Delta\theta)} [W^n_{j,m+1} - W^n_{j,m-1}], \tag{55}$$

and following the previous pattern,

$$\omega^2(n-1)\, W - rfe^{-i\omega t} \to \omega^2(n_{j,m} - 1)\, W^n_{j,m} - r_j f_{j,m} e^{-i\omega n\lambda h}, \tag{56}$$

where again, $\lambda = \Delta t/\Delta r$, $h = \Delta r$. Combining all of these expressions yields

$$W_{j+1,m}^{n+1} = W_{j-1,m}^{n+1} + aW_{j+1,m}^{n} - b_j W_{j,m}^{n} + cW_{j-1,m}^{n} + d_{j,m}W_{j,m+1}^{n} + e_{j,m}W_{j,m-1}^{n} - \lambda\omega^2 h^2 F_{j,m}^{n} \tag{57}$$

with

$$F_{j,m}^{n} = r_j(1 - n_{j,m})\, e^{i\omega[r_j\cos\theta_m - r_j - \lambda hn]}, \tag{58}$$

$$a = 1 + \lambda; \qquad c = \lambda - 1, \tag{59}$$

$$b_j = 2\lambda\{1 + \mu^2/r_j^2\} - \lambda\omega^2 h^2(n_{j,m} - 1), \tag{60}$$

$$d_{j,m} = \{\lambda\mu^2 + \tfrac{1}{2}\lambda\mu h \cot\theta_m\}/r_j^2, \tag{61}$$

$$e_{j,m} = 2\lambda\mu^2/r_j^2 - d_{j,m}, \tag{62}$$

and

$$\mu = \Delta r/\Delta\theta. \tag{63}$$

3.1. *The Reflecting Sphere*

In this section we solve Eq. (57) for the unit reflecting sphere problem. For this problem $r \in [1, R]$ and the step size $h = \Delta r$ is $(R - 1)/N$. Once again, N is an even integer. We set

$$r_1 = 1; \qquad r_j = 1 + (j - 1)h \qquad \text{for} \quad 1 \leqslant j \leqslant (N + 1) \tag{64}$$

and

$$r_{N+1} = R. \tag{64'}$$

Since the incident wave is symmetric about $\theta = 0$ and π, we only consider the range $\theta \in [0, \pi]$ (i.e., $W(r, \theta, t) = W(r, -\theta, t)$). Then $\Delta\theta = \pi/M$ and

$$\theta_1 = 0, \qquad \theta_m = (m - 1)\,\Delta\theta, \qquad 1 \leqslant m \leqslant M + 1, \qquad \theta_{M+1} = \pi. \tag{65}$$

Again at $r = R = r_{n+1}$ we have $W_r = 0$ and

$$W_{N,m}^{n} = W_{N+1,m}^{n}, \qquad n \geqslant 0, \quad 1 \leqslant m \leqslant M + 1. \tag{66}$$

The boundary condition (54) becomes

$$W_{1,m}^{n} = -\exp[i\omega(\cos\theta_m - n\lambda h - 1)]. \tag{67}$$

The initial condition is

$$W_{j,m}^{0} = Q_{j,m} \qquad \text{for some } Q(r, \theta). \tag{68}$$

For symmetry it is necessary that $Q(r, \theta) = Q(r, -\theta)$.

The sweeping method is again used to solve (57). But as we can see, a vertical sweep will give $W_{3,m}^{n}$ only for $2 \leqslant m \leqslant M$. We cannot let $m = M + 1$, since $\cot\theta_m = \infty$

there and $d_{j,m+1}$ becomes singular. This is where we apply the regularity condition (53). We set

$$W^n_{j,M+1} = W^n_{j,M}, \qquad 1 \leqslant j \leqslant N+1, \tag{69}$$

and

$$W^n_{j,1} = W^n_{j,2}, \qquad 1 \leqslant j \leqslant N+1. \tag{70}$$

(This also saves storage.)

The sweeping method coupled with (66), (69), and (70) yields the numerical solution.

3.2. *The Inhomogeneous Medium*

The scheme is the same as that in the previous case except for three points:

(1) $r \in [0, R]$, $h = R/N$, and $r_1 = 0$.

(2) $W^n_{1,m} = 0$, $n \geqslant 0$, $1 \leqslant m \leqslant M+1$.

We considered only cases where

(3) $n(r, \theta) = n(r, -\theta)$,

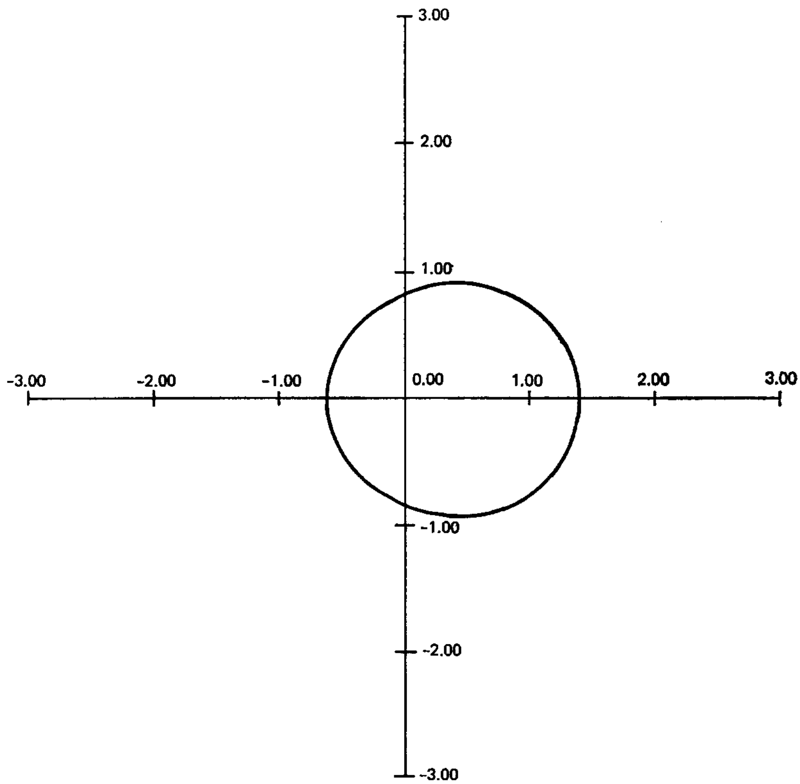

FIG. 3. The polar graph of the cross section, $F(\theta)$, for a metal sphere with $u \sim F(\theta)(e^{i\omega r}/r)$, $\omega = ka = 1$, $\Delta r = 0.1$, $\Delta\theta = \pi/40$, and $\Delta t = 1/250$.

so that W has the same symmetry. The solution of (57) follows on the interval $[0, \pi]$ by the method outlined in the case of the reflecting cylinder.

Once again, it is necessary to avoid the singularity at the origin by setting

$$W_{1,m}^n = W_{2,m}^n = W_{3,m}^n = 0; \qquad n \geqslant 0; \quad 1 \leqslant m \leqslant (M+1). \tag{71}$$

3.3. *Numerical Experiments*

We have run the program successfully only for a metal sphere. Again the classical method is to separate variables, but now one ends up with a Fourier series involving spherical Hankel and Bessel functions.

We have applied our method to this problem for the cases: $\omega = 1$ and $\omega = 5$. In both cases we have achieved good agreement with the tabulated results given by Bowman *et al.* [4]. These results are shown in Figs. 3 and 4. In both cases we had $h = 0.1$ and $N = 80$ with $\Delta\theta = \pi/40$ and $\Delta t = h/25$.

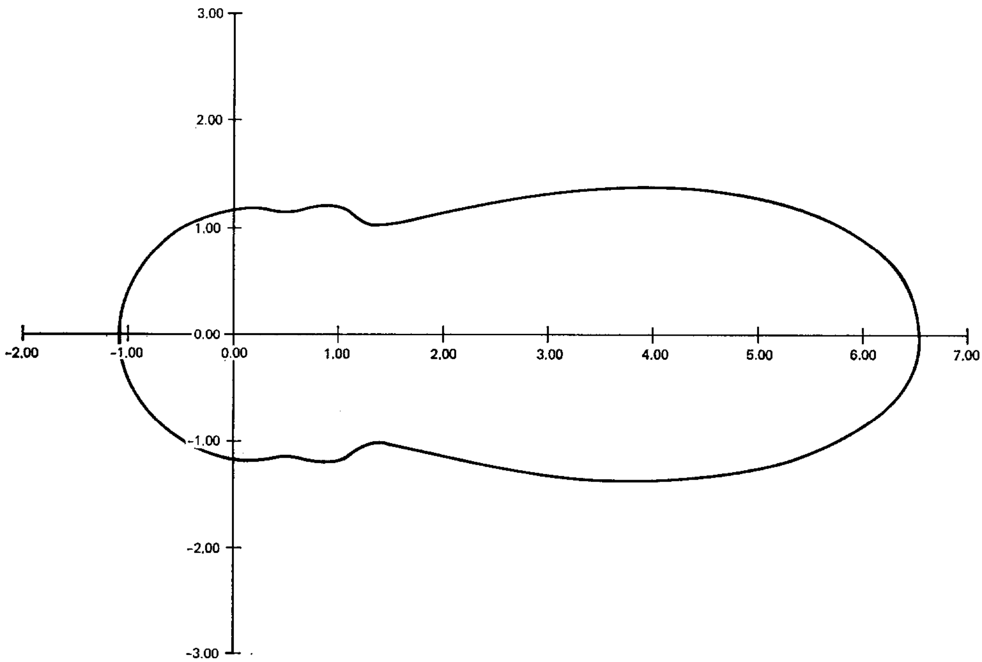

FIG. 4. Same as Fig. 3 with $\omega = 5$.

APPENDIX: SWEEPING OUT OF INITIAL DATA

We consider the solution of the simplest second-order equation

$$2W_{rt} = W_{rr} \tag{0.1}$$

in the strip $(0, 1) \times (0, \infty)$ and subject to the initial and boundary data

$$W(0, t) = W_r(1, t) = 0, \tag{0.2}$$

$$W(r, 0) = Q(r). \tag{0.3}$$

From (0. 1) we find

$$W_r = F(r + \tfrac{1}{2}t). \tag{0.4}$$

That is, W_r is a constant along the characteristics $r + \frac{1}{2}t = \text{constant}$. Therefore, along each characteristic that cuts the line $r = 1$ we have $W_r = 0$. Thus in the region $t > 2(1 - r)$, $W_r = 0$ or $W = h(t)$. But $W = 0$ for $r = 0$. Thus

$$W \equiv 0 \qquad \text{for} \quad t > 2. \tag{0.5}$$

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